Resistance distance*

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The theory of resistive electrical networks is invoked to develop a novel view: if fixed resistors are assigned to each edge of a connected graph, then the effective resistance between pairs of vertices is a graphical distance. Several theorems concerning this novel distance function are established.

1. Orientation

Professor Frank Harary has had a singular influence on graph theory through his own extensive original research, through the training of several researchers who themselves have made many important contributions, and through popularizing work, most significantly his text *Graph Theory* [1]. Frank Harary has contributed to a number of more advanced specialized texts, like that of Buckley and Harary [2] on graphical distance. What might well be *the* general lesson from all this work is: the possibilities for the development of the field of graph theory are ever open for innovative ideas.

An example exhibiting the openness of the field is found in the charming monograph by Doyle and Snell [3], wherein a novel probabilistic representation and consequent theorems for electric-circuit conduction problems are described. This work evidently builds from some initial mathematical work of Nash-Williams [4] in 1960, although the graph-theoretic study of electric circuits began over 150 years ago with Kirchhoff's analysis [5], with much modern work in electrical engineering.

Presumably, a qualitative new graphical distance would potentially be of great interest. Judging from the book *Distances in Graphs* by Buckley and Harary [2], earlier work has exclusively concerned one general type of graphical distance, where the distance between two sites of a graph is taken as the minimal sum of (positive) edge weights along a path between the two sites. The simplest "canonical" case takes each edge to be of unit weight, but more general weights should be of relevance in

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a great variety of circumstances. For example, in chemistry different multiplicity bonds are crucially distinct, so the associated edge weights might reasonably be taken to be ordered inversely to the bond multiplicities (or bond orders). *But* if multiple bonds indicate shorter distances, then multiple shortest paths between two more well-separated vertices might be anticipated to indicate a shorter "chemical distance" also. That is, for example , in fig. 1 the "effective distances" between a and b might be considered to decrease in going from graph G_1 to G_2 to G_3, even though all edge weights are taken to be unity. One might imagine that the "difficulty" of transport

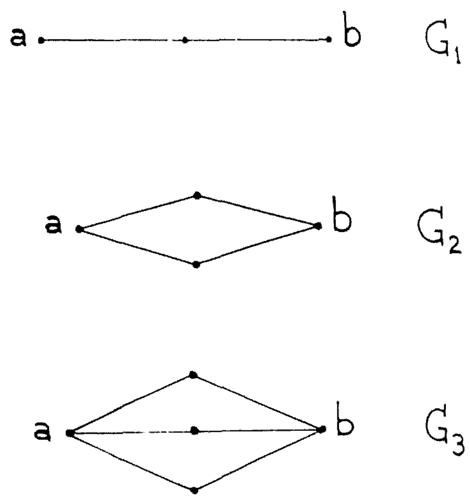

Fig. 1. Three graphs with the same conventional graphical distance between vertices a and b, although in successive graphs the "communication" between a and b might be imagined to be improved.

from a to b is a measure of distance and further that the "difficulty" increases the fewer the number of traffic routes between a and b. Indeed, even with two paths of different lengths one might imagine that the ease of communication between a and b might be enhanced somewhat by the longer path, so that the two sites could be viewed as closer than were they only connected by one of the two paths. With the allowance of such a mutual influence of multiple pathways, discontinuities in relevant paths might be avoided as one smoothly changes the weights so that the shortest pathway changes.

Here we propose a new distance function with the characteristic of multiple-route distance diminishment. Indeed, our approach is based on electrical network theory, wherein a fixed resistor is imagined on each edge. Then the distance between two vertices is defined as the (effective) resistance between the two nodes (when a battery is connected across them). Thence, to identify the distance between vertices a and b for graphs G_1, G_2 and G_3 of fig. 1, we imagine corresponding electrical

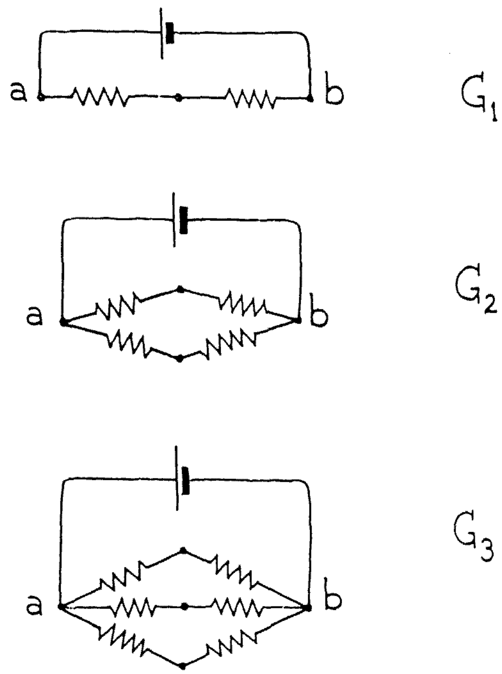

Fig. 2. The three graphs of fig. 1 with resistors (denoted by —ΛΛΛ—) introduced on each edge, while a battery (denoted by —|⊢—) is linked between the *a*, *b*-pair of vertices.

networks as in fig. 2. Thence, for all the resistors taking a value of 1 (ohm), one obtains (using standard series and parallel relations) respective resistance distances

$$
\begin{aligned}
\Omega_{ab} &= 1+1=2 && \text{for } G_1,\\
\Omega_{ab} &= 1/(\tfrac{1}{2}+\tfrac{1}{2})=1 && \text{for } G_2,\\
\Omega_{ab} &= 1/(\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2})=\tfrac{2}{3} && \text{for } G_3.
\end{aligned}
\tag{1.1}
$$

The distances between other pairs of vertices are obtained as effective resistances for other patterns of connection of the battery between other pairs of vertices, as in fig. 3. The distances (in ohm) here may be found, after some analysis, to be 1, 3/4 and 2/3. Notably, a distance between two vertices separated by two bonds is not necessarily the sum of the distances along these two intervening bonds.

Generally, for a battery delivering a current I the voltage (or potential across a and b) will be

$$v_{ab} = I\,\Omega_{ab}. \tag{1.2}$$

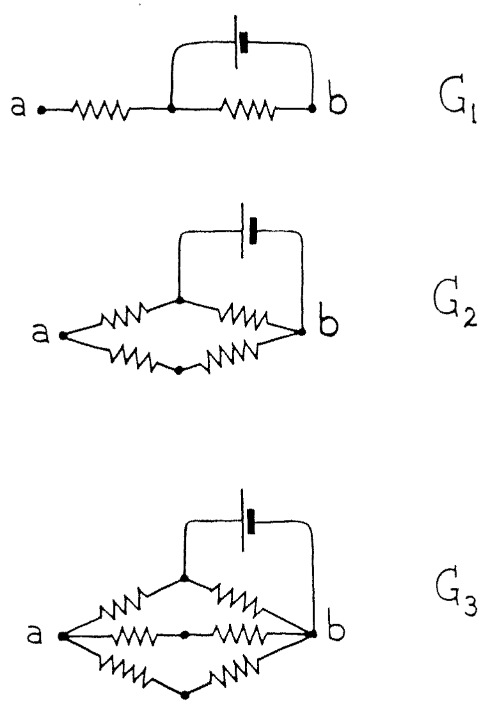

Fig. 3. The three graphs of fig. 1, again with resistors on each edge, but now with the battery connected to a different pair of vertices.

To determine Ω_{ij} for other pairs of vertices, the battery would be detached from a and b, then reattached between i and j. As an alternative to a battery, one could simply imagine using an ohm-meter. We assume our graphs are finite, and for the most part we follow standard [1] graph-theoretic nomenclature, e.g., $V(G)$ and $E(G)$ being the vertex and edge sets for graph G.

2. Background ideas

A formal presentation of some standard electrical network ideas [3,6] is of use in the following sections. A *G-flow* from vertex a to b of a graph G is defined to be a function i on pairs of adjacent sites such that

$$i_{xy} = -i_{yx} \tag{2.1}$$

and

$$\sum_{y}^{\sim x} i_{xy} = I\delta(x,a) - I\delta(x,b) \qquad x \in V(G), \tag{2.2}$$

where the sum is over $y \in V(G)$ adjacent to vertex x. Here, I is said to be the value of the net current out of the *source* site a and into the *sink* site b. Relation (2.2) is known as Kirchhoff's current-flow law (for the special case of a single source and sink). A G-flow is further said to be *physical* if there is an associated *potential* function υ on the vertices of G such that

$$i_{xy} r_{xy} = \upsilon_x - \upsilon_y \quad x, y \in E(G), \tag{2.3}$$

where $r_{xy} \equiv r_e$ if $e = \{x, y\}$. The potential function is also called the "voltage", although in a hydraulic framework it is called the "pressure". Relation (2.3) is known as *Ohm's law*. That for a G-flow there is such a potential may be shown to be equivalent to the requirement of Kirchhoff's *circuital law*,

$$\sum_{x \sim y}^{C} i_{xy} r_{xy} = 0 \quad \text{all } C, \tag{2.4}$$

where the sum is over the edges of a cycle C in G with the arguments of i ordered sequentially around C.

Associated to a graph G we identify a *normed space* with an orthonormal basis whose elements are in one-to-one correspondence with the vertices of G. Such a basis vector is denoted by $|x)$, $x \in V(G)$. There are several matrices (i.e. linear operators) of importance acting on this space. The *admittance* (or bond-order) matrix A has elements given as

$$A_{xy} = (x|A|y) = \begin{Bmatrix} 1/r_{xy} & x \sim y \\ 0 & \text{otherwise} \end{Bmatrix} \quad x, y \in V(G). \tag{2.5}$$

Clearly, for the standard choice of unit resistances, A reduces to the adjacency matrix of G. A second simple matrix also arises, namely the *degree* matrix Δ with elements

$$\Delta_{xy} = (x|\Delta|y) = \delta(x, y) \sum_{z}^{\sim x} 1/r_{xz}, \tag{2.6}$$

where the sum is over the $z \in V(G)$ that are adjacent to x. In the following section, the combination $\Delta - A$ plays a crucial role, and sometimes it is given a name, e.g. "Laplacian" matrix or "admittance" matrix (whence $\pm A$ is renamed as the "edge-admittance" or "mutual admittance" matrix). It is useful to have the following:

LEMMA 0

The matrix $\Delta - A$ has real eigenvalues, the minimum one of which is zero. If G is connected, this eigenvalue is nondegenerate and the associated eigenvector is (up to a scalar factor)

$$|\phi) \equiv \sum_{x} |x).$$

The first step of the proof, that the eigenvalues are real, is a well-known consequence of the fact that the matrix in question is Hermitean. Since all the off-diagonal elements of $\Delta - A$ are non-positive, the Frobenius–Perron theorem [7] implies that it has an eigenvector for the minimum eigenvalue such that all the nonzero components of the eigenvector may be chosen of like phase, the convenient choice being real positive. By application of $\Delta - A$ to the vector $|\phi)$, identified in the lemma statement, one has

$$(\Delta - A)|\phi) = 0, \tag{2.7}$$

so that $|\phi)$ is an eigenvector satisfying the criterion of non-negative components (here, all equal to 1). Since all eigenvectors to any other eigenvalue must (for a Hermitean matrix) be orthogonal to $|\phi)$, none of these others can share this non-negativity criterion, and 0 must be the minimum eigenvalue to $\Delta - A$. Finally, if G is connected, then for every $x, y \in V(G)$, one sees that $(x|(\Delta - A)^m|y) \neq 0$ for some (x, y-dependent) choice of m. Thence, a final part of the Frobenius–Perron theory [7] implies that this minimum eigenvalue occurs as a nondegenerate root of the secular polynomial, and the proof is complete.

As a consequence of this lemma, $\Delta - A$ has no inverse. However, within the subspace orthogonal to $|\phi)$, it does have an inverse. The matrix equal on this subspace to this inverse and otherwise being 0 is denoted by $Q/(\Delta - A)$, where Q is the (Hermitean and idempotent) *projection*

$$Q = \mathbf{1} - \frac{1}{(\phi|\phi)}\,|\phi)(\phi|. \tag{2.8}$$

This "resolvent" matrix $\{Q/(\Delta - A)\}$ satisfies

$$\begin{aligned} &\{Q/(\Delta - A)\}(\Delta - A) = (\Delta - A)\{Q/(\Delta - A)\} = Q, \\ &\{Q/(\Delta - A)\}Q = Q\{Q/(\Delta - A)\} = \{Q/(\Delta - A)\} \end{aligned} \tag{2.9}$$

and is called the *generalized* inverse of $\Delta - A$.

3. Effective resistance

We are now ready for a known [6] basic result:

LEMMA A

A physical G-flow from vertex a to b of a connected graph G exists, is unique, and is given by

$$i_{xy} = \frac{I}{r_{xy}}\,(x - y|Q/(\Delta - A)|a - b),$$

where $|a - b) \equiv |a) - |b)$.

To prove this, substitute from (2.3) into (2.2) to obtain

$$\sum_{y}^{\sim x} \frac{1}{r_{xy}} \{v_x - v_y\} = I\delta(x,a) - I\delta(x,b). \tag{3.1}$$

However, with the use of (2.6) and (2.5), this becomes

$$(x|\Delta|x)v_x - \sum_y (x|A|y)v_y = I(x|a-b). \tag{3.2}$$

Since this is true for arbitrary $x \in V(G)$, this may be recast as

$$(\Delta - A)\sum_x v_x|x) = I|a-b). \tag{3.3}$$

Then, because of lemma 0, this relation may be inverted on the subspace orthogonal to $|\phi)$ of (2.7). As a consequence,

$$\sum_x v_x|x) = I\{Q/(\Delta - A)\}|a-b) + c|\phi), \tag{3.4}$$

where c is an as yet undetermined constant. This yields directly the formula of our theorem, thereby establishing the uniqueness of these differences. The potential difference between a pair of vertices x, y is consequently given as

$$v_x - v_y = I(x-y|Q/(\Delta - A)|a-b). \tag{3.5}$$

Then, Ohm's law of (2.3) yields the formula of the lemma, the uniqueness of I, and its existence.

The potential difference between two points is seen in (3.6) to be directly proportional to I. For the choice $x = a$ *and* $y = b$, this proportionality constant is termed the effective *resistance* Ω_{ab} between a and b. Thence, we have the basic result of this section:

THEOREM A

For a physical G-flow from a to b,

$$\Omega_{ab} = (a-b|Q/(\Delta - A)|a-b).$$

The result of this theorem may be cast as a more conventional matrix equality if we introduce the diagonal matrix ∇ with elements

$$\nabla_{ab} \equiv \delta_{ab}(a|Q/(\Delta - A)|b). \tag{3.6}$$

Then a simple rearrangement of the result of the theorem gives

COROLLARY A

A graph G has a resistance matrix

$$\Omega = \nabla|\phi)(\phi| + |\phi)(\phi|\nabla - 2\{Q/(\Delta - A)\}.$$

As a consequence, all effective resistances are obtained via a matrix inversion. If desired, the generalized inverse $Q/(\Delta - A)$ may be computed in terms of an ordinary inverse: by finding the ordinary inverse to $\Delta - A + |\phi)(\phi|$, then subtracting $|\phi)(\phi|/(\phi|\phi)^2$.

For example, for the ("square") graph G_2 of fig. 1, we have (for $r = 1$ ohm)

$$\Delta - A = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix},$$

$$\frac{Q}{\Delta - A} = \frac{1}{16}\begin{bmatrix} 5 & -1 & -3 & -1 \\ -1 & 5 & -1 & -3 \\ -3 & -1 & 5 & -1 \\ -1 & -3 & -1 & 5 \end{bmatrix}, \qquad (3.7)$$

$$\Omega = \begin{bmatrix} 0 & 3/4 & 1 & 3/4 \\ 3/4 & 0 & 3/4 & 1 \\ 1 & 3/4 & 0 & 3/4 \\ 3/4 & 1 & 3/4 & 0 \end{bmatrix}.$$

The traditional "series" and "parallel" manipulations (alluded to in section 1) also serve in this special case to yield Ω rather directly.

4. Resistance is distance

In this section, we formally establish the identification of effective resistances as distances. By a *distance function* on G, we mean (as is standard [2]) a mapping ρ from the Cartesian product $V(G) \times V(G)$ to the real numbers such that the following axioms are satisfied:

$$\rho(b, a) \geq 0,$$
$$\rho(a, b) = 0 \Leftrightarrow a = b,$$
$$\rho(a, b) = \rho(b, a),$$
$$\rho(a, x) + \rho(x, b) \geq \rho(a, b), \qquad (4.1)$$

for any vertices $a, x, b \in V(G)$. The value $\rho(a, b)$ is said to be the *ρ-distance* (or distance, in abbreviated nomenclature) between a and b. The first two, the third and the fourth conditions of (4.1) are termed non-negativity, symmetric and triangle conditions, respectively.

Our fundamental result is:

THEOREM B

The resistance function on a graph is a distance function.

To begin the proof, we note that corollary A and the properties of the operator $\Delta - A$ as appear in lemma A yield the result that Ω_{ab} is symmetric and non-negative with $\Omega_{ab} = 0$ iff $a = b$. The focus of the proof then is the triangle inequality (on the last line of (4.1)). Let i and i' be G-flows from a to x and from x to b associated with potentials v and v', respectively. Then it is easily verified that

$$j \equiv i + i' \tag{4.2}$$

is an I-flow from a to b with associated potential

$$w = v + v'. \tag{4.3}$$

Now,

$$I\Omega_{ab} = w_a - w_b = \{v_a - v_b\} + \{v'_a - v'_b\}. \tag{4.4}$$

However, the extreme values of the potential v_y must be at $y = a$ and x, since otherwise some other more extreme site would be either a source or a sink. Likewise, v'_y is extreme at $y = x$ *and* b. Thence,

$$I\Omega_{ab} \le \{v_a - v_x\} + \{v'_x - v'_b\} = I\Omega_{ax} + I\Omega_{xb} \tag{4.5}$$

and the theorem follows.

5. Resistance sum rules

The general result of this section is:

THEOREM C

If G is a connected graph and Z is an arbitrary symmetric matrix, then

$$\sum_{a,b} (b|(\Delta - A)Z(\Delta - A)|a)\Omega_{ab} = 2\,\mathrm{tr}(\Delta - A)Z.$$

To prove this, abbreviate $(\Delta - A)Z(\Delta - A)$ to X and use theorem A to obtain

$$\sum_{a,b} (b|X|a)\Omega_{ab} = 2\sum_{a,b} (b|X|a)\{(a|Q/(\Delta - A)|a) - (a|Q/(\Delta - A)|b)\}. \tag{5.1}$$

The right-hand side of this equation yields two double-sum terms, the first of which entails a factor

$$\sum_{b}(b|X|a) = (\phi|(\Delta - A)Z(\Delta - A)|a) = 0, \tag{5.2}$$

where we have recalled the eigenvector $|\phi)$ of lemma 0. Thence,

$$\begin{aligned}\sum_{a,b}(b|X|a)\Omega_{ab} &= -2\sum_{a,b}(b|(\Delta - A)Z(\Delta - A)|a)(a|\frac{Q}{\Delta - A}|b)\\ &= -2\,\mathrm{tr}(\Delta - A)Z(\Delta - A)\,\frac{Q}{\Delta - A}\\ &= -2\,\mathrm{tr}(\Delta - A)Z,\end{aligned} \tag{5.3}$$

which is the desired result.

This sum rule for resistances may be viewed as more special than theorem A; the present theorem has the feature that it avoids the (generalized) inverse of $\Delta - A$. One special choice for Z is as $Q/(\Delta - A)$, whence via (2.9) we obtain a result noted earlier by Weinberg [8]:

COROLLARY C1

For a connected graph,

$$\sum_{a,b}(a|A|b)\Omega_{ab} = 2(|V(G)| - 1).$$

A whole sequence of rules is obtained by taking Z as $(\Delta - A)^n$;

COROLLARY C2

For a connected graph

$$\sum_{a,b}(a|(\Delta - A)^n|b)\Omega_{ab} = -2\,\mathrm{tr}(\Delta - A)^n,$$

with n a non-negative integer.

For more highly symmetric graphs, these two corollaries yield nearer-neighbor effective resistances:

COROLLARY C3

For $e \in E(G)$ of an edge-transitive graph

$$\Omega_e = \frac{|V(G)| - 1}{|E(G)\ |}\ r,$$

where r is the internal resistance common to all edges.

COROLLARY C4

For a vertex- and edge-transitive graph such that all paths of length 2 are equivalent, the effective resistance between two next-nearest neighbor *nnn* sites is

$$\Omega_{nnn} = \frac{2}{d-1}\left\{1 - \frac{2}{|V(G)|}\right\} r,$$

where d is the common vertex degree.

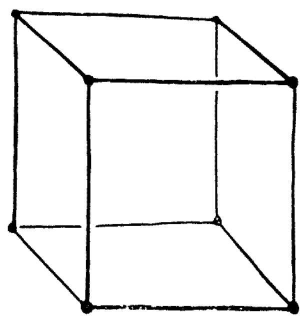

Fig. 4. The cube graph, upon each edge of which one may imagine a resistor r.

As an example, one might consider the cubic graph (of fig. 4) with equal resistors r on each edge. Then,

$$\begin{aligned} \Omega_e &= \frac{8-1}{12} r = \frac{7r}{12}, \\ \Omega_{nnn} &= \frac{2}{2}\left\{1 - \frac{2}{8}\right\} r = \frac{3r}{4}. \end{aligned} \tag{5.4}$$

Returning to corollary C2 with $n = 2$, after some manipulation one can even obtain the remaining resistance of $5r/6$.

6. Comparison

First, we note an intuitively appealing result (which Doyle and Snell [3] refer to as "Rayleigh's Monotonicity Law"):

LEMMA D

The resistance Ω_{ab} is a nondecreasing function of the edge resistances. This function is constant only for those edges not lying on any path between a and b.

The proof of this first statement may start with theorem A to take a derivative of Ω_{ab} with respect to a general edge resistance r_{xy},

$$\frac{\partial \Omega_{ab}}{\partial r_{xy}} = \{i_{x,y}/I\}^2 \geq 0. \tag{6.1}$$

The second part is more delicate, but may be approached in the case of no xy-containing path between a and b by G finding a cut-point to separate G into two pieces, one G_{xy} containing x and y, while the other contains a and b, and then show via Kirchhoff's laws that there is no current flow anywhere in G_{xy} and $i_{xy} = 0$. The result is proved in more detail (via another approach) in the publication by Doyle and Snell [3].

The conventional type of *graphical distance* between vertices a and b of G is [2]

$$D_{ab} \equiv \min_{\pi} \sum_{e \in \pi} \frac{1}{r_e}, \tag{6.2}$$

whence the minimum is taken over all paths π from a to b, and the sum is over all edges of π. We have:

THEOREM D

For all distinct pairs of vertices a, b in G, $D_{ab} \geq \Omega_{ab}$, with equality iff there is but a single path between a and b.

For the proof, let π be a path which gives a minimum sum in (6.2). Now consider how Ω_{ab} changes as resistances are increased, denoting the initial value by Ω_{ab}. Increasing a resistance r_e for any edge e not in π does not increase Ω_{ab}, as is seen from lemma D. Further, this lemma implies that if e is in any other path between a and b, then there is a strict decrease in Q_{ab}. Taking all $r_e \to \infty$ for all e not in π, one finally obtains $\Omega_{ab} \to D_{ab}$. However, then $\Omega_{ab} \geq D_{ab}$ with equality only as indicated in the theorem.

Recalling the standard result that there is a single unique path between any two points of a *tree*, one immediately has:

COROLLARY D

The conventional and resistance distances are the same between every pair of vertices of a connected graph iff the graph is a tree.

Thence, earlier results already found in the case of trees for D apply equally for Ω. See, for example, ref. [9].

7. An analogue theorem

One might anticipate some results which hold for the conventional graphical distance function D_{ab} to hold also for Ω_{ab}. Indeed, we note here one such analogue to a result due to Graham et al. [9]. However, first we start with what is in essence the standard result for "series" resistances:

LEMMA E

Let x be a cut-point of a commerical graph, and let a and b be points occurring in different components which arise upon deletion of x. Then,

$$\Omega_{ab} = \Omega_{ax} + \Omega_{xb}.$$

The proof may be briefly indicated if we consider the assumptive circumstances as indicated in fig. 5. If vertex a is the source of current I, then since sink b is not

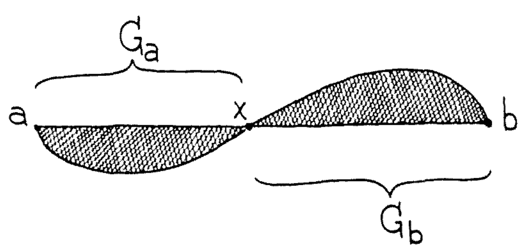

Fig. 5. The general form of the graph assumed for theorem D. Note that x but nothing to the right is included in G_a, whereas x but nothing to the left is included in G_b.

in the part G_a, all the current from a must pass through x, so that in the G_a portion, x acts as a sink with

$$v_{ax} = I\Omega_{ax}. \tag{7.1}$$

Further, since the net current into x is 0, the current leaving x into part G_b must be I, whence one is led to

$$v_{xb} = I\Omega_{xb}. \tag{7.2}$$

Addition of these two potential differences gives

$$v_{ab} = v_{ax} + v_{xb} = I(\Omega_{ax} + \Omega_{xb}), \tag{7.3}$$

whereupon one obtains the theorem.

Of course, there are other possible standard results [6] for parallel resistors (or star–triangle transformations), but our sought after "analogue" distance result can be obtained using just lemma E and a few definitions. Let cof $\mathcal{M}$ denote the sum of the co-factors (of the determinant) of a square matrix $\mathcal{M}$. Also, a *block* of a graph is defined to be a maximal subgraph without cut-points. Further, here we label resistance distance matrices by the graph to which they are associated. Thence, in analogy to the result [10] for $D(G)$:

THEOREM E

If G is a connected graph with blocks G_α, then

$$\operatorname{cof} \Omega(G) = \prod_{\alpha} \operatorname{cof} \Omega(G_\alpha),$$

$$\det \Omega(G) = \sum_{\alpha} \det \Omega(G_\alpha) \prod_{\beta}^{\neq\alpha} \operatorname{cof} \Omega(G_\beta).$$

The proof exactly follows that for the conventional graphical distance matrix $D(G)$ [10] . The crucial property required (beyond that of being a distance function) is that of lemma E.

8. Analogue definitions

Various quantities already introduced with regard to the conventional distance matrix D might also be introduced for Ω. Associated to the distance matrix Ω, there is a characteristic polynomial

$$\omega(x) \equiv \det(x\mathbf{1} - \Omega), \tag{8.1}$$

which we call the *resistance distance polynomial*. For the case of the graphical distances of the preceding section, such a polynomial has already found much use [2,9,11], so that the presently defined quantity should also be of interest.

The so-called *Wiener index* has found much use in chemistry, as reviewed elsewhere [12]. Originally, it was defined [13] for trees as

$$W = \sum_{a<b} D_{ab}, \tag{8.2}$$

but for trees $D_{ab} = \Omega_{ab}$ (as noted in corollary D), so that an extension to other connected graphs could be

$$W' \equiv \sum_{a<b} \Omega_{ab}, \tag{8.3}$$

although the usual extension has been via (8.2). We have a convenient general formula:

THEOREM F

For a connected graph with N vertices,

$$W' \equiv N \ \mathrm{tr}\{Q/(\Delta - A\}.$$

It is a simple matter of algebra to obtain

$$W' = \frac{1}{2}\sum_{a,b}(a-b|Q/(\Delta - A)|a-b) = N \ \mathrm{tr}\{Q/(\Delta - A\} - 2(\phi|Q/(\Delta - A)|\phi). \quad (8.4)$$

However, since $Q/(\Delta - A)$ is null on the $|\phi)$-space one immediately obtains the theorem.

9. Prospects

What we believe is that a novel and fundamental distance function on graphs has been identified. That its properties are already widely investigated in other contexts (both in electrical engineering and in mathematics) seems but an indication of the fundamental mathematical nature of this resistance distance function. The results developed here are intended to indicate some first mathematical features of resistance distance. The possibility for chemical utility seems to us likely since, first, the conventional distance has found several uses and second, the resistance distance has "multiple-route distance diminishment" features which we have already indicated (in section 1) and should have chemical relevance. Professor Harary has often noted for novel graph-theoretical concepts that further work seems warranted, which we believe is also the case here.

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